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Spectral collocation solutions to multiparameter Mathieu's system

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ABSTRACT

Our main aim is the accurate computation of a large number of specified eigenvalues and eigenvectors of Mathieu's system as a multiparameter eigenvalue problem (MEP). The reduced wave equation, for small deflections, is solved directly without approximations introduced by the classical Mathieu functions. We show how for moderate values of the cut-off collocation parameter the QR algorithm and the Arnoldi method may be applied successfully, while for larger values the Jacobi–Davidson method is the method of choice with respect to convergence, accuracy and memory usage.

1. Introduction

Mathieu's system is often used in the literature as a motivating example for the introduction of multiparameter eigenvalue problems (MEPs), see, for example, the monograph of Volkmer [1]. This MEP approach of this well-known system is a natural one. The system is obtained when separation of variables is applied to solve the vibration of a fixed elliptic membrane, see, for instance, the classical book by Meixner and Schäfke [2, Section 4.31].

However, to the best of our knowledge, the accurate numerical solution of Mathieu's system as a two-parameter eigenvalue problem was never studied in detail. The two-parameter dependence makes computing Mathieu's functions more involved than, for example, Bessel's functions.

Ruby [3] provides some examples from science and technology arguing that they deserve accurate solutions of Mathieu's system. From Igbokoyi and Tiab [4], as well as the references therein, it is apparent that the case of an ellipse with the minor axis approaching zero is of tremendous importance in petroleum engineering.

The main aim of this paper is to find a large number (say more than four hundred) of even and odd, π and 2π, eigenfrequencies and eigenmodes of Mathieu's system as a MEP *very accurately*. Up to our knowledge no one considered yet to compute hundreds of such eigenvalues. Neither the radial Mathieu's equation nor the Mathieu's system, as a two-parameter eigenvalue problem, have been solved using the Chebyshev collocation (pseudospectral) method. This approach, in conjunction with various methods to solve the (discretized) algebraic MEP, is considered in this paper. Particular attention is given to these methods, as well as to the sensitivity of the eigenvalues. For small to moderate values of the cut-off collocation parameter N, the QR algorithm and shift-and-invert Arnoldi method work satisfactory. For larger N they are too costly. The remedy is a Jacobi–Davidson based method which solves these cases accurately and efficiently.

In fact, the literature concerning the numerics of the second problem is rather poor. Neves [5] provides some numerical results along with a Klein oscillation theorem for the multiparameter Mathieu's system. These numerical results came from

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an *ad hoc* method. It involves a shooting scheme based on the Runge–Kutta method used to solve a two-point boundary value problem. Troesch and Troesch [6] find the two lowest eigenvalues using the Bessel functions for the representation of Mathieu's functions. Gutiérrez-Vega and coauthors [7] use the Fourier representation to find classical Mathieu's functions. Without the need of special functions, Wilson and Scharstein [8] use a Fourier collocation method to find a "wide range" of eigenfrequencies, i.e., the first hundred modes. Instead of solving a MEP, they solve a sequence of two generalized eigenvalues and this seems to affect the accuracy of the obtained solutions.

In contrast, the angular Mathieu's equation is solved by Trefethen [9] and Weideman and Reddy [10] by Fourier collocation; this is thoroughly analyzed by Boyd in his monograph [11]. More recently, Shen and Wang [12] provide approximation results (in Sobolev spaces) for the eigenmodes of the first Mathieu equation. They also solve the second Mathieu equation by a spectral Galerkin method and eventually by a Legendre spectral-Galerkin method they solve a Helmholtz and a modified Helmholtz equation.

The paper is organized as follows. In Section 2 we introduce the Mathieu system as a MEP, i.e., the four possible differential MEPs. We comment on the two-parameter algebraic eigenvalue problems in Section 3 and provide an overview of the Jacobi–Davidson method to solve such problems in Section 4. The Chebyshev collocation discretization as well as a finite difference discretization of the Mathieu's system as a MEP is considered in Section 5. Our numerical results are presented in Section 6. Some conclusions can be found in Section 7.

2. Mathieu's system

The coupled system of Mathieu's angular and radial equations in which a and q are independent parameters will be thought of now as a multiparameter (two-parameter) eigenvalue problem. The following four MEPs can be formulated with respect to Mathieu's system:

- a π-even problem

$$\begin{cases} G''(\eta) + (a - 2q\cos(2\eta))G(\eta) = 0, & 0 < \eta < \frac{\pi}{2}, \\ G'(0) = G'(\frac{\pi}{2}) = 0, & \\ F''(\xi) - (a - 2q\cosh(2\xi))F(\xi) = 0, & 0 < \xi < \xi_0, \\ F\prime(0) = F(\xi_0) = 0, & \end{cases} \tag{1}$$

- a 2π-even problem

$$\begin{cases} G''(\eta) + (a - 2q\cos(2\eta))G(\eta) = 0, & 0 < \eta < \frac{\pi}{2}, \\ G'(0) = G(\frac{\pi}{2}) = 0, & \\ F''(\xi) - (a - 2q\cosh(2\xi))F(\xi) = 0, & 0 < \xi < \xi_0, \\ F'(0) = F(\xi_0) = 0, & \end{cases} \tag{2}$$

- a π-odd problem

$$\begin{cases} G''(\eta) + (a - 2q\cos(2\eta))G(\eta) = 0, & 0 < \eta < \frac{\pi}{2}, \\ G(0) = G(\frac{\pi}{2}) = 0, & \\ F''(\xi) - (a - 2q\cosh(2\xi))F(\xi) = 0, & 0 < \xi < \xi_0, \\ F(0) = F(\xi_0) = 0, & \end{cases} \tag{3}$$

- a 2π-odd problem

$$\begin{cases} G''(\eta) + (a - 2q\cos(2\eta))G(\eta) = 0, & 0 < \eta < \frac{\pi}{2}, \\ G(0) = G'(\frac{\pi}{2}) = 0, & \\ F''(\xi) - (a - 2q\cosh(2\xi))F(\xi) = 0, & 0 < \xi < \xi_0, \\ F(0) = F(\xi_0) = 0. & \end{cases} \tag{4}$$

These coupled systems of two-point boundary value problems come from the problem of a vibrating elliptic membrane Ω with fixed boundaries $\partial\Omega$,

$$(\Delta + \omega^2)\psi(x,y) = 0, \quad (x,y) \in \Omega, \quad \psi(x,y) = 0, \quad (x,y) \in \partial\Omega, \tag{5}$$

when the separation of the variables, i.e., $\psi(x,y) := F(\xi)G(\eta)$ is used in the elliptical coordinates ξ and η,

$$x := h\cosh(\xi)\cos(\eta),$$
$$y := h\sinh(\xi)\sin(\eta), \quad 0 \leqslant \xi < +\infty, \quad 0 \leqslant \eta < 2\pi.$$

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Thus

$$\xi_0 := \operatorname{arccosh}(\frac{\alpha}{h}),$$

where $h := \sqrt{\alpha^2 - \beta^2}$ is half the distance between the foci of the membrane. The parameter q is related to the eigenfrequency ω by

$$q := \frac{h^2\omega^2}{4}$$

and a is the parameter arising in the separation of variables. Volkmer analyzes in his monograph [1] the above four systems in detail. For these right definite MEP, he provides results concerning the existence and countability of eigenvalues, numbers of zeros of eigenfunctions and the completeness of even and odd sets of eigenmodes. His analytical results are exhaustive. A Klein oscillation theorem is also available in [5] and some other useful comments on the formulations above can be found in [13]. The Mathieu system can also be "embedded" in the most general setting of the multiparameter eigenvalue problem for ordinary differential equations formulated by Sleeman in [14].

3. Algebraic two-parameter eigenvalue problem

An *algebraic two-parameter eigenvalue problem* has the form

$$\begin{cases} A_1 x = \lambda B_1 x + \mu C_1 x, \\ A_2 y = \lambda B_2 y + \mu C_2 y, \end{cases} \tag{6}$$

where A_i, B_i, and C_i are given $n_i \times n_i$ complex matrices, $\lambda, \mu \in \mathbb{C}$, $x \in \mathbb{C}^{n_1}$, and $y \in \mathbb{C}^{n_2}$. A pair (λ, μ) is called an *eigenvalue* if it satisfies (6) for nonzero vectors x and y. The tensor product $x \otimes y$ is then the corresponding *eigenvector*.

The two-parameter eigenvalue problem (6) can be expressed as two coupled generalized eigenvalue problems as follows. On the tensor product space $S := \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ of the dimension $m := n_1 n_2$ we define so called *operator determinants*

$$\Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2,$$
$$\Delta_1 = A_1 \otimes C_2 - C_1 \otimes A_2,$$
$$\Delta_2 = B_1 \otimes A_2 - A_1 \otimes B_2$$

(for details on the tensor product and relation to the multiparameter eigenvalue problem, see, for example, [15]). The two-parameter eigenvalue problem (6) is *nonsingular* when Δ_0 is nonsingular. In this case the matrices $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute and (6) is equivalent to a coupled pair of generalized eigenvalue problems

$$\begin{cases} \Delta_1 \mathbf{z} = \lambda \Delta_0 \mathbf{z}, \\ \Delta_2 \mathbf{z} = \mu \Delta_0 \mathbf{z} \end{cases} \tag{7}$$

for decomposable tensors $\mathbf{z} = x \otimes y \in S$ (see [15]).

There exist some numerical methods for two-parameter eigenvalue problems. If m is small, we can apply the existing numerical methods for the generalized eigenvalue problem to solve the coupled pair (7). An algorithm of this kind, which is based on the QZ algorithm, is presented in [16].

When m is large, it is not feasible to compute all eigenpairs. There are some iterative methods that can be applied to compute some solutions. Most of them require good initial approximations to avoid misconvergence. One such method, that we apply in our numerical experiments, is the Tensor Rayleigh Quotient Iteration (TRQI) from [17], which is a generalization of the standard Rayleigh quotient iteration, (see, e.g., [18]). This method computes one eigenpair at a time.

In case when we are interested in more than just one eigenpair and we do not have any initial approximations, a method of choice is a Jacobi–Davidson type method [16]. The state-of-the-art, which uses harmonic Ritz values [19], can be used to compute a small number of eigenvalues of (6), which are closest to a given target $(\lambda_T, \mu_T) \in \mathbb{C}^2$. A brief overview of the method is presented in the next section.

4. Overview of Jacobi–Davidson method

For the numerical solution we exploit a Jacobi–Davidson method as developed in [16,19,20]. In this method the eigenvectors x and y are sought in search spaces $\mathcal{U}$ and $\mathcal{V}$, respectively. There are two main phases: expansion of the subspaces, and extraction of an approximate eigenpair from the search space. First consider the subspace expansion. Suppose that we have approximate eigenvectors $u \approx x$ and $v \approx y$ with corresponding approximate eigenvalue $(\sigma, \tau) \approx (\lambda, \mu)$; for instance, the tensor Rayleigh quotient. We are interested in orthogonal improvements $s \perp u$ and $t \perp v$ such that

$$A_1(u+s) = \lambda B_1(u+s) + \mu C_1(u+s), \tag{8}$$
$$A_2(v+t) = \lambda B_2(v+t) + \mu C_2(v+t). \tag{9}$$

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Let

$$\mathbf{r}_1 = (A_1 - \sigma B_1 - \tau C_1)\,u,$$
$$\mathbf{r}_2 = (A_2 - \sigma B_2 - \tau C_2)\,v$$

be the residuals of vector $u \otimes v$ and value (σ, τ). We can rewrite (8) and (9) as

$$(A_1 - \sigma B_1 - \tau C_1)s = -\mathbf{r}_1 + (\lambda - \sigma)B_1 u + (\mu - \tau)C_1 u + (\lambda - \sigma)B_1 s + (\mu - \tau)C_1 s, \tag{10}$$

$$(A_2 - \sigma B_2 - \tau C_2)t = -\mathbf{r}_2 + (\lambda - \sigma)B_2 v + (\mu - \tau)C_2 v + (\lambda - \sigma)B_2 t + (\mu - \tau)C_2 t. \tag{11}$$

We neglect the second-order correction terms $(\lambda - \sigma)B_1 s + (\mu - \tau)C_1 s$ and $(\lambda - \sigma)B_2 t + (\mu - \tau)C_2 t$. Let $V \in \mathbb{R}^{(n_1+n_2)\times 2}$ be a matrix with columns (for reasons of stability, preferably orthonormal) such that

$$\operatorname{span}(V) = \operatorname{span}\left(\begin{bmatrix} B_1 u \\ B_2 v \end{bmatrix}, \begin{bmatrix} C_1 u \\ C_2 v \end{bmatrix}\right)$$

and let $W \in \mathbb{R}^{(n_1+n_2)\times 2}$ be

$$W = \begin{bmatrix} u & \mathbf{0} \\ \mathbf{0} & v \end{bmatrix}.$$

With the oblique projection

$$P = I - V(W^T V)^{-1} W^T$$

onto $\operatorname{span}(V)^{\perp}$ along $\operatorname{span}(W)$, it follows that

$$P\mathbf{r} = \mathbf{r} \quad \text{and} \quad P\begin{bmatrix} B_1 u \\ B_2 v \end{bmatrix} = P\begin{bmatrix} C_1 u \\ C_2 v \end{bmatrix} = \mathbf{0},$$

where $\mathbf{r} = [\mathbf{r}_1\ \mathbf{r}_2]^T$. Therefore, we can project out the first-order terms $(\lambda - \sigma)B_1 u + (\mu - \tau)C_1 u$ and $(\lambda - \sigma)B_2 v + (\mu - \tau)C_2 v$ using this oblique projection, reformulating (10) and (11) (without the neglected second-order correction terms) as

$$P\begin{bmatrix} A_1 - \sigma B_1 - \tau C_1 & 0 \\ 0 & A_2 - \sigma B_2 - \tau C_2 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} = -\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} \tag{12}$$

for $s \perp u$ and $t \perp v$. We use (possibly inexact) solutions s and t to this linear system to expand the search spaces $\mathcal{U}$ and $\mathcal{V}$.

Now we focus on the subspace extraction. As introduced in [19], the harmonic Rayleigh–Ritz extraction for the MEP extracts approximate vectors $u,\ v$ and corresponding values σ and τ by imposing the Galerkin conditions

$$\begin{aligned} A_1 u - \sigma B_1 u - \tau C_1 u \quad &\perp \quad (A_1 - \sigma B_1 - \tau C_1)\mathcal{U}, \\ A_2 v - \sigma B_2 v - \tau C_2 v \quad &\perp \quad (A_2 - \sigma B_2 - \tau C_2)\mathcal{V}. \end{aligned} \tag{13}$$

This generally turns out to be a method of choice for interior eigenvalues near a target $(\sigma,\ \tau)$. A basic pseudocode for the method is given in Algorithm 1, where RGS stands for repeated Gram–Schmidt, or any other numerically robust method to expand an orthonormal basis.

Algorithm 1. A Jacobi–Davidson type method for the MEP

Input: Starting vectors u, v and a tolerance ε
Output: An approximate eigenpair (θ, η, u, v) for the MEP
$s = u,\ t = v,\ U_0 = [\,],\ V_0 = [\,]$
for $k = 1, 2, \ldots$
  RGS $(U_{k-1}, s) \to U_k$, RGS $(V_{k-1}, t) \to V_k$
  Extract an approximation (u, v, θ, η) using (13)
  Solve (approximately) $s \perp u, t \perp v$ from (12)
end

5. Discretization of Mathieu's system as a MEP

Our previous numerical experiments concerning non-standard, high order and singularly perturbed eigenvalue problems reported in [21–23] proved that the Chebyshev collocation method is fairly accurate, flexible and implementable. It turned out to be superior to the spectral Galerkin or tau method also based on the Chebyshev polynomials. The well-known monograph of Fornberg [24] provides a thorough analysis with how, when and why this pseudospectral approach works.

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Thus, the Chebyshev collocation discretization of MEP (1) reads

$$\begin{cases} \left(\left(\frac{4}{\pi}\right)^2 \cdot {}^{e,\pi}D_{\mathrm{n}}^2 + (a - 2q \cdot \mathrm{diag}(\cos(\pi(x_C+1)/2)))\right)u = \mathbf{0}, \\ \left(\left(\frac{2}{\xi_0}\right)^2 \cdot {}^{e,\pi}D_{\mathrm{nd}}^2 - (a - 2q \cdot \mathrm{diag}(\cosh(\xi_0(x_C+1))))\right)v = \mathbf{0}, \end{cases} \tag{14}$$

where ${}^{e,\pi}D_{\mathrm{n}}^2$ and ${}^{e,\pi}D_{\mathrm{nd}}^2$ are second order differentiation matrices in the Chebyshev nodes of the second kind x_C, i.e.,

$$x_C := \left\{\cos\left(\frac{(k-1)\pi}{N-1}\right),\ k = 1, 2, \ldots, N\right\}.$$

In the symbol ${}^{e,\pi}D_{\mathrm{n}}^2$ the upper indices e and π stand for even and π period, and the lower index n for the Neumann boundary conditions

$$G'(0) = G'\left(\frac{\pi}{2}\right) = 0,$$

which are enforced. Similarly, in ${}^{e,\pi}D_{\mathrm{nd}}^2$ the mixed boundary conditions

$$F'(0) = F(\xi_0) = 0,$$

are introduced, so n comes from the first and d from the second boundary condition. We use the seminal paper of Weideman and Reddy [10] to obtain the entries of these two matrices and the simple and general strategy of Hoepffner [25] to impose all boundary conditions. These matrices are also available in the book of Trefethen [9]. The vectors u and v contain the unknown values of G and F in the nodes x_C.

Thus, the problem (14) is an algebraic MEP of type (6) with (a, q) standing for (λ, μ). The discretizations for the last three problems (2)–(4) are analogous.

Unfortunately, the matrices ${}^{e,\pi}D_{\mathrm{n}}^2$ and ${}^{e,\pi}D_{\mathrm{nd}}^2$ are dense, non-symmetric and have high condition numbers (see, for instance, [23]).

The pseudospectra (see [26] for definition and numerical code) of even problems (1) and (2) are depicted in Fig. 1. This picture shows mildly sensitive eigenvalues with decreasing sensitivity for large $a(q)$.

It is worth noting at this moment that the curves $a(q)$ represent the solutions of the first Sturm–Liouville problems in (1) *and* (2) for $q \in [0, 10]$. They are the interlaced quasi "vertical" curves in Fig. 1. The family of curves $A(q)$ depicts the solutions of the second Sturm–Liouville problem in (1) *or*(2) for the same range of q. They are represented by the quasi "oblique" curves. Their intersections localize the eigenpairs (a, q) of MEP (1). Our Fig. 1 refines Fig. 1 from Neves [5].

The Chebyshev collocation discretization of MEP (4) reads

$$\begin{cases} \left(\left(\frac{4}{\pi}\right)^2 \cdot {}^{o,2\pi}D_{\mathrm{dn}}^2 + (a - 2q \cdot \mathrm{diag}(\cos(\pi(x_C+1)/2)))\right)u = \mathbf{0}, \\ \left(\left(\frac{2}{\xi_0}\right)^2 \cdot {}^{o,2\pi}D_{\mathrm{dn}}^2 - (a - 2q \cdot \mathrm{diag}(\cosh(\xi_0(x_C+1))))\right)v = \mathbf{0}. \end{cases}$$

In ${}^{o,2\pi}D_{\mathrm{dn}}^2$ the upper index o stands for odd property, the index 2π for period and dn for the mixed boundary conditions

$$G(0) = G'\left(\frac{\pi}{2}\right) = 0.$$

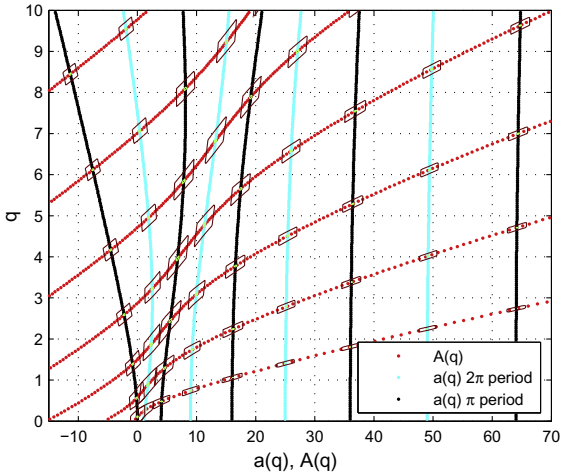

Fig. 1. The overlapped pseudospectra of problems (1) and (2), $N = 24$, $\alpha = \cosh(2)$, $\beta = \sinh(2)$.

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The matrix ${}^{o,2\pi}D^2_{\mathrm{dn}}$ with the lower index d involves the symmetric Dirichlet boundary conditions

$$F(0) = F(\xi_0) = 0.$$

The pseudospectra of odd problems (3) and (4), using equal levels as in Fig. 1, are depicted in Fig. 2.

As we see, the eigenvalues are even less sensitive than those of (1) and (2). A possible explanation is the fact that the Dirichlet boundary conditions

$$F(0) = F(\xi_0) = 0$$

in (3) and (4) induce a sort of symmetry in the differentiation matrices.

To evaluate the performances of our strategy we carried out numerical experiments on the finite difference discretization of our differential eigenvalue problems. Thus the usual finite difference of (1) reads

$$\begin{cases} \left(\left(\frac{2}{\pi}\right)^2 \cdot {}^{e,\pi}D^{2,FD}_{\mathrm{n}} + (a - 2q \cdot \mathrm{diag}(\cos(\pi x_{N+1})))\right) u = \mathbf{0}, \\ \left(\left(\frac{1}{\frac{\xi_0}{\pi}}\right)^2 \cdot {}^{e,\pi}D^{2,FD}_{\mathrm{nd}} - (a - 2q \cdot \mathrm{diag}(\cosh(2\xi_0 x_N)))\right) v = \mathbf{0}, \end{cases} \tag{15}$$

where ${}^{e,\pi}D^{2,FD}_{\mathrm{n}}$ and ${}^{e,\pi}D^{2,FD}_{\mathrm{nd}}$ stand for the second order centered finite difference approximation of the second derivative in the $N+1$ equispaced nodes x_{N+1}.

The matrices ${}^{e,\pi}D^{2,FD}_{\mathrm{n}}$ and ${}^{e,\pi}D^{2,FD}_{\mathrm{nd}}$ are now symmetric and tridiagonal of order $N+1$ and N respectively and the Neumann boundary conditions were introduced by *mirror imaging technique* described in the monograph of Quarteroni, Sacco, and Saleri [27, p. 549]. Despite these simplifications in (15) the numerical results provided in the next section are obviously inferior to those obtained by the Chebyshev collocation (see Table 3 and Table 4).

It is important to point out at this moment that Wilson and Scharstein [8] use a Fourier collocation (and not Galerkin) method to discretize Mathieu's system. Their shape (trial) functions are some trigonometric functions which implicitly satisfy the boundary conditions but it is not clear from this paper what is the distribution of their nodes and how they are clustered to the boundary. As this paper is detailed in the monograph [28] it seems that a uniform grid is used. Our strategy takes the advantage of the Chebyshev clustering to the boundary.

6. Numerical examples

In our numerical experiments we compute solutions of Mathieu's systems (1)–(4) using discretizations from Section 5. For each of the four systems we know from Volkmer [1] that for every pair of nonnegative indices (i,j) there exists a pair (a_{ij}, q_{ij}) with nonzero functions F_{ij} and G_{ij} such that G_{ij} has exactly i zeros on $(0, \frac{\pi}{2})$ and F_{ij} has exactly j zeros on $(0, \xi_0)$. This is one way how we can index the solutions.

Another option of indexing comes from the fact that Mathieu's systems (1)–(4) are related to the problem of a vibrating elliptic membrane with fixed boundaries (5). Each solution (a,q) gives an eigenmode of (5) with the eigenfrequency $\omega = 2\sqrt{q}/h$. The solutions of (1) and (2) give all even eigenmodes of (5). We order the even eigenmodes so that $\omega^e_1 \leqslant \omega^e_2 \leqslant \cdots$. To each even eigenmode (see, for example, [5] or [8]) we can associate an index (k,l), where k is the number of zeros of G on $(0,\pi)$, and l is the number of zeros of F on $(0,\xi_0)$. The eigenmode is then $\psi^{k,l}_e(x,y) = F(\xi)G(\eta)$. In a similar way the solutions of (3) and (4) give the odd eigenmodes $\psi^{k,l}_o$ of (5).

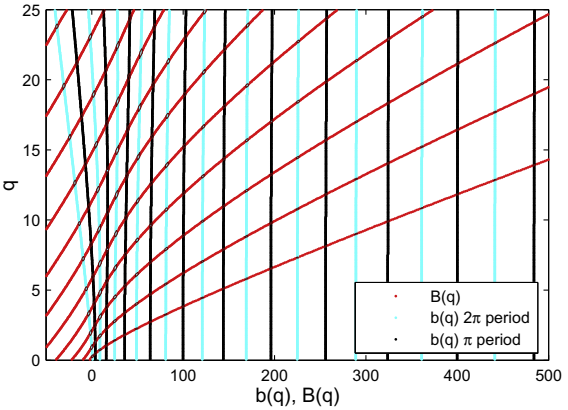

Fig. 2. The overlapped pseudospectra of problems (3) and (4), $N = 24$, $\alpha = \cosh(2)$, $\beta = \sinh(2)$.

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In particular, if F_{ij} and G_{ij} are solutions of one of Mathieu's systems (1)–(4), then

$$F_{ij}(\xi)G_{ij}(\eta) = \begin{cases} \psi_e^{2i,j}(x,y) & \text{for (1)},\\ \psi_e^{2i+1,j}(x,y) & \text{for (2)},\\ \psi_o^{2i+2,j}(x,y) & \text{for (3)},\\ \psi_o^{2i+1,j}(x,y) & \text{for (4)}. \end{cases}$$

The choice of the method to solve the algebraic MEP's depends on the requested eigenvalue and the required accuracy. It is clear that if we want to compute a higher eigenfrequency very accurately, we need a larger N. Depending on the size of N, we propose to use one of the following methods:

(a) EIG-Γ: When N is small, we can apply existing numerical methods (for instance `eig` in Matlab) to the eigenvalue problem

$$\Delta_0^{-1}\Delta_2 \boldsymbol{z} = \mu \mathbf{z}, \tag{16}$$

where the matrix $\Gamma_2 := \Delta_0^{-1}\Delta_2$ is of size $N^2 \times N^2$; we note that Δ_0 is a diagonal matrix. The obtained eigenvector $\mathbf{z}$ is decomposable, i.e., $\mathbf{z} = x \otimes y$, and it is easy to compute x and y from $\mathbf{z}$.

(b) EIGS-Γ: When matrix Γ_2 is too large for (a), we can apply the implicit shift-and-invert Arnoldi (available as function `eigs` in Matlab) to (16). The matrix Γ_2 is quite sparse: it has N full blocks of size $N \times N$ on its diagonal, whereas all non-diagonal $N \times N$ blocks are diagonal matrices. In many cases, when we need just a small number of eigenvalues, (b) is more efficient than (a) even for a small N.
If N is very large, this approach is no longer feasible. The first problem is that the L and u factors of the LU decomposition of the matrix $\Gamma_2 - \sigma I$ are virtually full triangular matrices, and we run out of memory.
Although we could try to use another solver instead of the default LU decomposition in `eigs`, there is another problem when N is large. Namely, as the matrix Γ_2 has size $N^2 \times N^2$, the method builds its search space by vectors of size N^2, which is time and memory consuming.

(c) JD-W: When N is too large for (b), we can apply the Jacobi–Davidson method. An advantage of the Jacobi–Davidson method is that it works with matrices and vectors of size N. Therefore, the method might be applied when (b) is too expensive.

The results were obtained using Matlab R2011b running on Intel Core Duo P8700 2.53 GHz processor using 4 GB of memory. In this environment, the approach EIGS-Γ works up to $N = 80$, for larger N we have to use JD-W. The method EIGS-Γ might be more efficient than EIG-Γ if many eigenvalues are required. Matlab implementations of the algorithms are available on e-mail request.

Example 1. We compare `EigElip`, which is a Matlab implementation of our method EIGS-Γ, to the Matlab function `runelip` by Wilson [29], which was used to compute the eigenfrequencies in [8].

Table 1 contains the results for the computation of the n lowest even eigenfrequencies for the ellipse with given α and β. Parameters N_1 and N_2 for `EigElip` specify the number of points used for the discretization of Mathieu's systems, which might be different for each of the two equations. The number N_1 is used for the angular equation and N_2 is used for the radial equation. The values are chosen so that the computed eigenfrequencies are correct to at least 10 decimal places. The parameter nrts $= (k_m, l_m)$ in `runelip` specifies that the method computes all eigenfrequencies of index (k,l) where $k \leqslant k_m$ and $l \leqslant l_m$. The values are minimal possible so that all of the n lowest eigenfrequencies are among the computed ones. The

Table 1
Comparison of `EigElip` and `runelip`.

α	β	n	`EigElip`			`runelip`		
			(N_1, N_2)	Time	Error	Nrts	Time	Error
2	1	100	(54, 25)	2.5	3e−11	(26,6)	10.3	3e−11
2	1	200	(66, 32)	8.5	2e−11	(38,9)	23.6	5e−11
2	1	300	(80, 36)	23.2	3e−11	(48,11)	37.7	6e−11
2	1	400	(85, 40)	42.0	5e−11	(56,13)	54.0	6e−11
2	1	500	(93, 45)	78.4	3e−11	(63,14)	70.0	3e−01
4	1	100	(68, 24)	3.5	5e−11	(35,5)	12.5	2e−11
4	1	200	(86, 26)	10.2	5e−11	(50,6)	22.1	3e−11
4	1	250	(94, 28)	16.1	3e−11	(56,7)	29.0	3e−06
4	1	300	(100, 30)	24.8	5e−11	(62,8)	36.6	3e−03
8	1	100	(84, 18)	3.1	2e−11	(48,3)	11.1	2e−11
8	1	125	(94, 20)	5.3	2e−11	(55,4)	17.2	3e−05
8	1	150	(100, 20)	6.9	1e−11	(60,4)	19.4	1e−02
cosh(2)	sinh(2)	100	(39, 43)	3.7	2e−11	(24,9)	10.0	1e−11

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computational times, which are given in seconds, show that the new method is considerably faster than `runelip` for a modest n. For large n `runelip` can be faster than `EigElip` (see $\alpha = 2$, $\beta = 1$, and $n = 500$), but also less accurate. The values in the 6th and the 9th column present the maximum absolute error of the computed eigenvalues, where the "exact" eigenvalues to compare with were computed with larger N_1 and N_2. One can see that `runelip` becomes inaccurate for higher eigenfrequencies, in particular when the ratio α/β is large (see also Example 4).

Example 2. In this example we use Jacobi–Davidson with harmonic Ritz values, presented in Section 4, to compute eigenvalues close to a given target. Depending on the region of interest we do this for several targets. The results of this phase is a set of eigenpairs $((\lambda_k, \mu_k), x_k \otimes y_k)$ for $k = 1, \ldots, m$. For each obtained eigenvalue (λ_k, μ_k) we compute its index (i_k, j_k), where i_k and j_k are the number of zeros of x_k and y_k, respectively. Here we assume that vectors x_k and y_k are discrete approximations of continuous curves.

In the second phase we extend the obtained set by the TRQI. We exploit the following property of eigenvectors of Mathieu's system. Let $x_1 \otimes y_1$ and $x_2 \otimes y_2$ be approximate eigenvectors belonging to the eigenvalues with indices (i_1, j_1) and (i_2, j_2), respectively. If $j_1 = j_2$ and i_1 is close to i_2, then x_1 and x_2 do not differ much. The same applies to y_1 and y_2 when $i_1 = i_2$ and j_1 is close to j_2. This is displayed on Fig. 3, where the x part of the eigenvector corresponding to the eigenvalue with index $(4, j)$ is presented for $j = 2, \ldots, 6$. So, for each pair of eigenvectors, such that i_1 is close to i_2 and j_1 is close to j_2, we can apply TRQI with an initial approximation $x_1 \otimes y_2$ to compute the eigenpair with the index (i_1, j_2). This simple approach usually converges in a couple of steps.

We take the Chebyshev discretization (14) with matrices of size 100×100 and two targets: $(0, 0)$ and $(100, 0)$. For each target we do 200 outer iterations of the Jacobi–Davidson method with harmonic Ritz values. As preconditioners we take $(A_i - \sigma_T B_i - \tau_T C_i)^{-1}$ for $i = 1, 2$, where (σ_T, τ_T) is the current target. We apply 20 steps of the GMRES to solve the correction equations. As a result, we get 44 eigenpairs for the target $(0, 0)$ and 27 eigenpairs for the target $(100, 0)$. From these eigenpairs we compute additional 13 eigenvalues with the TRQI, so that in the end we have approximations for all eigenpairs of (1) with indices (i, j), such that $i \leqslant 13$ and $j \leqslant 5$.

The computed eigenvalues are presented on Fig. 4 with different symbols. The eigenvalues marked by dot and $\times$-mark were computed using the Jacobi–Davidson method with targets $(0, 0)$ and $(100, 0)$, respectively, while the eigenvalues marked by plus were computed using the TRQI. One can see that in a large majority the eigenvalues computed by the Jacobi–Davidson method are indeed the closest ones to the given target.

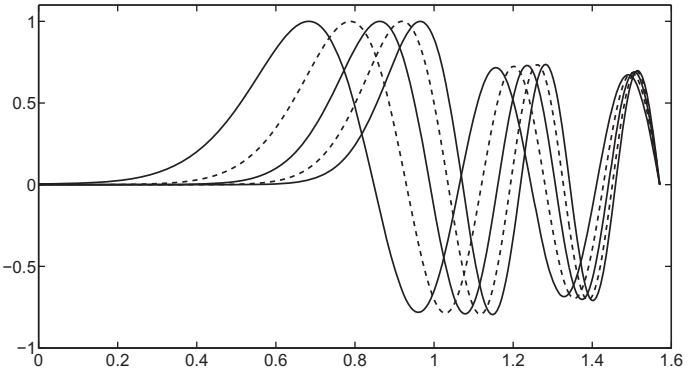

Fig. 3. x-Part of eigenvector of (14) corresponding to the eigenvalue with index $(4, j)$ for $j = 2, \ldots, 6$.

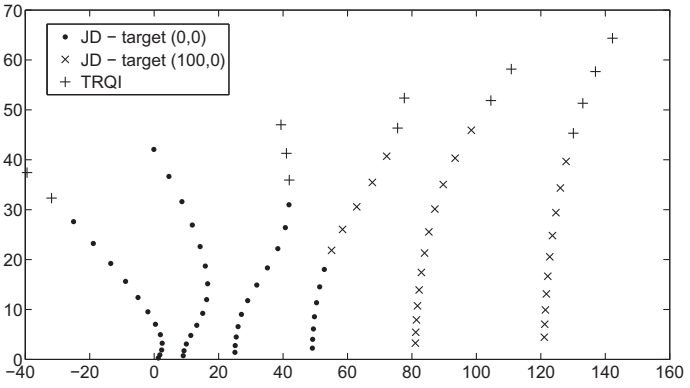

Fig. 4. All eigenvalues of (1) with indices (i, j) for $i \leqslant 13$ and $j \leqslant 5$, computed by the Jacobi–Davidson method using targets $(0, 0)$ and $(100, 0)$, and extended by the TRQI.

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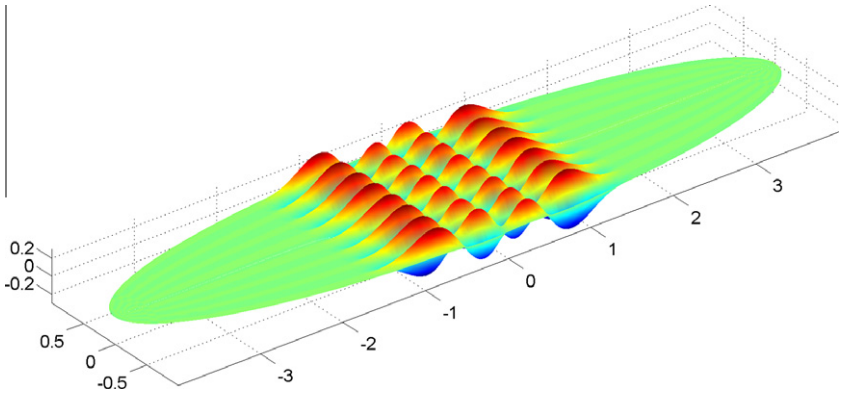

Fig. 5. Eigenmode $\psi_e^{3,8}$ for the ellipse with $\alpha = 4$ and $\beta = 1$.

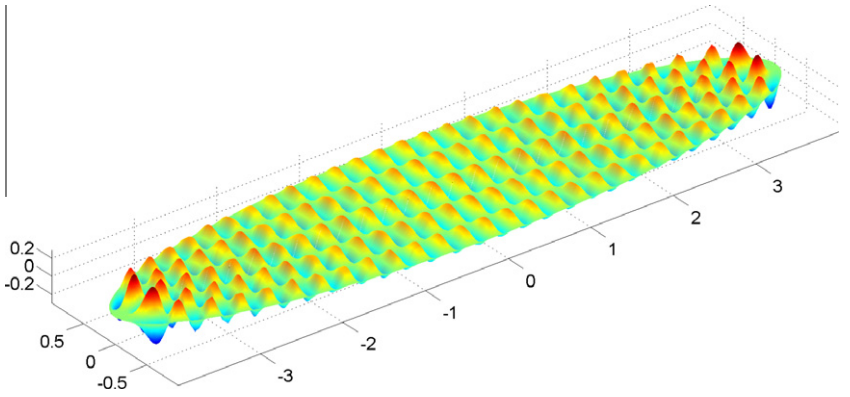

Fig. 6. Eigenmode $\psi_e^{52,3}$ for the ellipse with $\alpha = 4$ and $\beta = 1$.

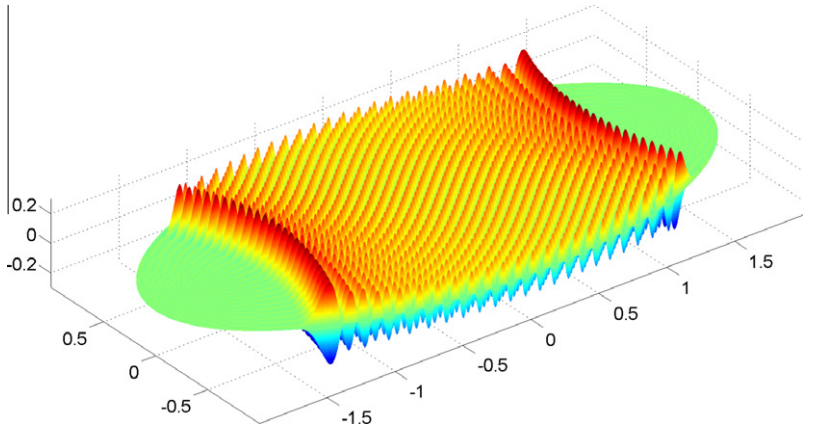

Fig. 7. Eigenmode $\psi_e^{41,25}$ for the ellipse with $\alpha = 2$ and $\beta = 1$ that has its eigenfrequency closest to $\omega = 100$.

Example 3. Using the method EIGS-Γ we can accurately compute higher eigenmodes than the previously reported in the literature.

For example we take $\alpha = 4$ and $\beta = 1$ and compute the lowest 300 even eigenmodes using `EigElip` with $N_1 = 120$ and $N_2 = 40$. The eigenmodes $\psi_e^{3,8}$ and $\psi_e^{52,3}$ for the eigenfrequencies $\omega_{298}^e = 24.45490912$ and $\omega_{300}^e = 24.53067377$ are presented in Figs. 5 and 6, respectively.

It is important to underline that even higher eigenmodes, which require larger matrices, could not be obtained by EIG-Γ and EIGS-Γ methods due to memory limitations, while JS-W is able to compute these eigenmodes up to the required accuracy.

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Table 2
The lowest 10 even eigenfrequencies for the ellipse with $\alpha = 1000$ and $\beta = 1$.

n	Eigenfrequency	n	Eigenfrequency
1	1.57129649	6	1.57630126
2	1.57229680	7	1.57730317
3	1.57329744	8	1.57830539
4	1.57429840	9	1.57930793
5	1.57529967	10	1.58031079

Table 3
Accuracy of the Chebyshev collocation.

N	Error for ω_1^e	Error for ω_{50}^e	Error for ω_{100}^e
(20,10)	$1.8 \cdot 10^{-10}$	$1.4 \cdot 10^{-2}$	$2.8 \cdot 10^{-3}$
(30,15)	$8.1 \cdot 10^{-14}$	$5.9 \cdot 10^{-6}$	$1.4 \cdot 10^{-4}$
(40,20)	$9.5 \cdot 10^{-14}$	$3.0 \cdot 10^{-10}$	$8.0 \cdot 10^{-8}$
(50,25)	$2.1 \cdot 10^{-13}$	$1.2 \cdot 10^{-14}$	$1.2 \cdot 10^{-11}$

Table 4
Accuracy of finite differences.

N	Error for ω_1^e	Error for ω_{50}^e	Error for ω_{100}^e
200	$5.9 \cdot 10^{-3}$	$5.7 \cdot 10^{-2}$	$3.9 \cdot 10^{-2}$
400	$3.0 \cdot 10^{-3}$	$2.6 \cdot 10^{-2}$	$1.2 \cdot 10^{-2}$
800	$1.5 \cdot 10^{-3}$	$1.2 \cdot 10^{-2}$	$4.5 \cdot 10^{-3}$
1600	$7.4 \cdot 10^{-4}$	$6.0 \cdot 10^{-3}$	$1.8 \cdot 10^{-3}$

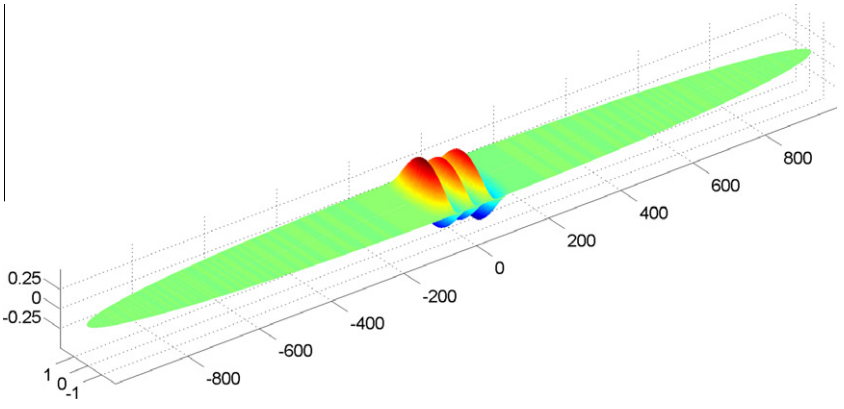

Fig. 8. Eigenmode $\psi_e^{6,1}$ for the ellipse with $\alpha = 1000$ and $\beta = 1$.

Using $N_1 = N_2 = 500$, the corresponding Γ_2 matrix (that we do not compute explicitly) has dimension 250000×250000. For the target $(1, 7500)$ we computed the even eigenfrequency of the ellipse with $\alpha = 2$ and $\beta = 1$ that is closest to the target $\omega_T = 100$. The result is $\omega = 99.97702290$. Its eigenmode $\psi_e^{41,25}$ is presented on Fig. 7.

Example 4. There are certain applications that require the eigenmodes of an ellipse with a very large ratio α/β; see, for instance, [4].

In this example we show that such problems can also be solved efficiently via the MEP approach. If we take the ellipse with $\alpha = 1000$ and $\beta = 1$, and set $N_1 = 200$ and $N_2 = 15$, then `EigElip` returns the 10 lowest even eigenfrequencies in Table 2. The 6th lowest even eigenmode is shown in Fig. 8.

Let us remark that `runelip` fails to compute the 10 lowest eigenfrequencies. It returns 0, followed by 4 eigenfrequencies smaller than 1. Such results provide confidence in the validity of our approach.

Example 5. We compare the accuracy of the computed eigenfrequencies if we discretize the Mathieu's system (1) by the Chebyshev collocation discretization (14) or by the standard finite differences (15). We take $\alpha = 2$, $\beta = 1$ and compute even eigenfrequencies ω_1^e, ω_{50}^e, and ω_{100}^e. The absolute errors for the Chebyshev collocation and for the finite differences are

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collected in Tables 3 and 4, respectively, where the "exact" eigenvalues were computed by the Chebyshev collocation using larger N_1 and N_2. It is obvious that in spite of better conditioned matrices involved (symmetric, tridiagonal, etc.) finite differences require much larger matrices to obtain accurate results. On the other hand, using the Chebyshev collocation we can compute eigenvalues quite accurately with relatively small matrices.

It is worth noting that the largest Γ_2 matrix corresponding to finite differences discretization has dimension $1600^2 \times 1600^2$ and the eigenvalue problem can only be solved using JD algorithm.

7. Conclusions

In this paper we have accurately solved Mathieu's system as a MEP for domains with geometrical aspects ranging from a circle to extremely flattened ellipse, e.g., a ratio of the major to minor axes of ellipses of 10^3. This means that the method is stable with respect to the geometry (eccentricity) of the problem.

Accurate numerical computation of high frequencies is much harder than for low frequencies. We introduced a hierarchy of numerical methods that can deal with the corresponding algebraic eigenvalue problems for increasing N, and we are able to compute eigenfrequencies and the corresponding eigenmodes from the first ones to the order of about 10^4. However, the accuracy varies from a quasi spectral one for the lowest mode to a moderate one for the highest mode.

With respect to the accuracy as well as to the time required, our algorithm is superior to those reported in literature. It is also stable with respect to the degree N of the spectral approximation, as is apparent from our reported numerical experiments.

All in all, our new algorithm can be used to solve the MEP associated to Mathieu's system corresponding to a large variety of geometrical settings.

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